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Quantum phase transition in the generalized single-mode superradiant model

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Abstract. In this paper we reveal a zero-temperature quantum phase transition for the single-mode superradiant model with the form A^2 from the normal to superradiant phase by mean of the Holstein-Primakoff transformation. In the thermodynamic limit, in which the numbers of atoms becomes infinite, the ground state energy and corresponding wavefunctions of both the normal and superradiant phases are obtained and therefore the scaling behavior near the critical transition point is derived.

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1 Introduction

It is well-known that the single-mode superradiant model (SMSM) [1] describes a collection of N two-level atoms interacting with a single bosonic mode via a dipole interaction with an atom-field and now belongs to essential ingredients in theoretical quantum optics [2], nuclear physics [3], quantum chaos [4], and quantum dissipation [5] etc. In this model the spin algebra can be used to describe single or many two-level atoms. A common feature of SMSM is that in general they are nonintegrable, with exact solutions available only for very specific cases [6]. One is simplified "dephasing models", where the spin couples to both the boson and static field via only one of its components [3]. The other is in the large spin limit where bosonic representations, which is an early example for using the Holstein-Primakoff (HP) transformation [7]. Dicke ever used this model to illustrate the importance of collective effects of superradiance, where the atomic ensemble spontaneously emits, as one would expect if the atoms were radiating incoherently.

The thermodynamic phase transition for SMSM was firstly investigated in rotating-wave approximation by Hepp and Lieb [8] and by Wang and Hioe through making use of a set of Glauber's coherent states for the field [9]. Based on reference [9], Hioe discussed a generalized SMSM including an interaction energy of the form $a^+\sigma^+ + a\sigma^-$ [10]. In the recent year, the quantum phase transition (QPT) for SMSM has been revealed by Emary and Brandes by using HP transformation [11,12] and then has been generalized to the case with arbitrary boson coupling in the pseudo-spin $x-z$ plane [13].

It has been clear that Hamiltonian for SMSM originates from the Hamiltonian describing that many atoms interact the electromagnetic fields with the assumptions as follows: the energy of the term A^2 is negligibly small; all the atoms vector potential can be evaluated at the center common in the long-wavelength limit; stat ψ is metastable and decay processes to lower states can be ignored; and under rotating-wave approximation, the antiresonant terms are allowed to omit. It should be noted that the controversy whether the thermodynamic phase transition can be affected by the term A^2 has been discussed in the finite temperature [14–21].

In contrast to these earlier works, in this paper we will consider this phase transition at zero temperature, where increasing the coupling parameter derives the system to undergo a transition from the normal to the super-radiant phases. In thermodynamic limit that $N \to \infty$, the corresponding ground state energy and wavefunctions of both the normal and super-radiant phase are given by means of HP transformation and therefore the scaling behavior near the critical transition point is also obtained. It is shown that by using this method the wavefunctions can provide a useful way to describe the quantum phase transition.

2 The SMSM in the HP representation

The generalized Hamiltonian of SMSM with the energy of the form $\kappa(a^+ + a)^2$ without the rotating-wave approximation is written as [14–21]

$$H = \omega a^+ a + \kappa(a^+ + a)^2 + \sum_{i=1}^{N}[\omega_0\sigma_z^i + \frac{\lambda}{\sqrt{N}}(\sigma_+^i + \sigma_-^i)(a^+ + a), \quad (1)$$

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where a, a^+ are the photon annihilation and creation operators; σ_+ and σ_- are the spin operators for the jth atom defined as $\sigma_\pm = \sigma_x \pm i\sigma_y$, where σ_x and σ_y are the x and y components of the Pauli matrices; λ measures the coupling of the interaction terms $(\sigma_+ + \sigma_-)(a^+ + a)$; κ describes the effect of the energy of the term $(a^+ + a)^2$; $\hbar\omega_0$ is the energy difference between the two levels of the atom and ω is the frequency of the electromagnetic wave; and N is the total number of atoms. The origin of the factor $1/\sqrt{N}$ in the interaction is the fact that the original dipole coupling strength is proportional to $1/\sqrt{V}$, where V is the volume of the cavity. If introduced the density of the atoms in the cavity $\rho = N/V$, it becomes $\sqrt{\rho/N}$ and by subsuming the density into the coupling constant we obtain $1/\sqrt{N}$.

In terms of the collective atomic operators defined as: $J_z = \sum_{i=1}^{N} \sigma_z^i$, $J_\pm = \sum_{i=1}^{N} \sigma_\pm^i$ with $j = N/2$, equation (1) can be written as

$$H = \omega a^+ a + \kappa(a^+ + a)^2 + \omega_0 J_z + \frac{\lambda}{\sqrt{2j}}(J_+ + J_-)(a^+ + a). \tag{2}$$

It is clear that the introduced collective atomic operators satisfy $SU(2)$ commutation relations such that $[J_z, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_z$. By using the HP transformation of the angular momentum operators defined as [7]: $J_+ = b^+\sqrt{2j - b^+b}$, $J_- = \sqrt{2j - b^+b}\,b$, $J_z = (b^+b - j)$, where $[b, b^+] = 1$, Hamiltonian (2) can be rewritten as

$$H = \omega_0(b^+b - j) + \omega a^+ a + \kappa(a^+ + a)^2 + \lambda(a^+ + a)(b^+\sqrt{1 - b^+b/2j} + \sqrt{1 - b^+b/2j}\,b). \tag{3}$$

Hamiltonian (3) contains virtually the formation of QPT in the thermodynamic limit, in which the number of atoms becomes infinite, $N \to \infty$, and therefore $j \to \infty$. Some detailed discussions will be given in the following sections.

3 The normal phase

If simply neglecting terms with j in the denominator of Hamiltonian (4) we can obtain effective Hamiltonian in this phase

$$H^{(N)} = \omega_0 b^+b + \omega a^+ a + \kappa(a^+ + a)^2 + \lambda(a^+ + a)(b^+ + b) - \omega_0 j, \tag{4}$$

which is a bilinear Hamiltonian in the bosonic operators and can be easily diagonalized by introduction of position and momentum operators such that: $a^+ = (\sqrt{\omega}x - i\,(1/\sqrt{\omega})\,p_x)/\sqrt{2}$ and $b^+ = (\sqrt{\omega_0}y - i\,(1/\sqrt{\omega_0})\,p_y)/\sqrt{2}$. Therefore we have

$$H^{(N)} = \frac{1}{2}\left[(\omega^2 + 4\kappa\omega)x^2 + p_x^2 + \omega_0^2 y^2 + p_y^2 + 4\lambda\sqrt{\omega\omega_0}xy - \omega - \omega_0\right] - j\omega_0. \tag{5}$$

By rotating the coordinate system defined as $x = q_1\cos\gamma^{(N)} + q_2\sin\gamma^{(N)}$ and $y = -q_1\sin\gamma^{(N)} + q_2\cos\gamma^{(N)}$ with $\tan(2\gamma^{(N)}) = 4\lambda\sqrt{\omega\omega_0}/[\omega_0^2 - (\omega^2 + 4\kappa\omega)]$, Hamiltonian (5) can be diagonalized as

$$H^{(N)} = \frac{1}{2}\left[\left(\varepsilon_-^{(N)}\right)^2 q_1^2 + p_1^2 + \left(\varepsilon_+^{(N)}\right)^2 q_2^2 + p_2^2 - \omega - \omega_0\right] - j\omega_0, \tag{6}$$

where

$$\left(\varepsilon_\pm^{(N)}\right)^2 = \frac{1}{2}\left\{\omega_0^2 + (\omega^2 + 4\kappa\omega) \pm \sqrt{[\omega_0^2 - (\omega^2 + 4\kappa\omega)]^2 + 16\lambda^2\omega\omega_0}\right\}. \tag{7}$$

It is important that $\varepsilon_-^{(N)}$ is real only when

$$\lambda \preceq \frac{1}{2}\sqrt{\omega_0(\omega + 4\kappa)} = \lambda_c, \tag{8}$$

where λ_c is nothing but the critical transition point between the normal and super-radiant phases. The fundamental excitations of the system are given by the energies $\varepsilon_\pm$ which describe collective modes where ε_- and ε_+ are recognized as the photonic and atomic modes respectively seen from their properties of λ-dependence (see Fig. 3).

In this phase, the ground state energy is given by

$$E_G^{(N)} = -j\omega_0, \tag{9}$$

which is $O(j)$, whereas the excitation energies $\varepsilon_\pm^{(N)}$ are $O(1)$. This means that by scaling our energies with j, the excitation spectrum above the ground state becomes quasicontinuous in the $j \to \infty$.

Since the normalized ground state wavefunctions of a single harmonic oscillator in terms of its coordinate q is a Gaussian, namely,

$$G_\pm^{(N)} = \left(\frac{\varepsilon_\pm^{(N)}}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{\varepsilon_\pm^{(N)}}{2}q^2\right),$$

in the $q_1 - q_2$ representation the ground state wavefunctions $\Psi_G^{(N)}$ are given by

$$\Psi_G^{(N)} = G_-^{(N)}(q_1)G_+^{(N)}(q_2). \tag{10}$$

In the $x - y$ representation, we have

$$\Psi_G^{(N)} = G_-^{(N)}\left(x\cos\gamma^{(N)} - y\sin\gamma^{(N)}\right) \times G_+^{(N)}\left(x\sin\gamma^{(N)} + y\cos\gamma^{(N)}\right). \tag{11}$$

From the numerical results illustrated by equation (11), Figures 2 show that the ground state wavefunctions are varied through the coupling parameter λ from 0 to 0.5269 with $\omega_0 = 1$, $\omega = 0.9$ and $\kappa = 0.053$. From Figure 2 it can be seen that the ground state wavefunctions are the product of orthogonal Gaussians of equal width at $\lambda = 0$; with the increasing of coupling parameter λ, the wave packet becomes stretched in a direction determined by the angle $\gamma^{(N)}$ that on resonance is simply equal to $\pi/4$; this stretching increases up to $\lambda_c = 0.527$, where the ground state wavefunctions diverge.

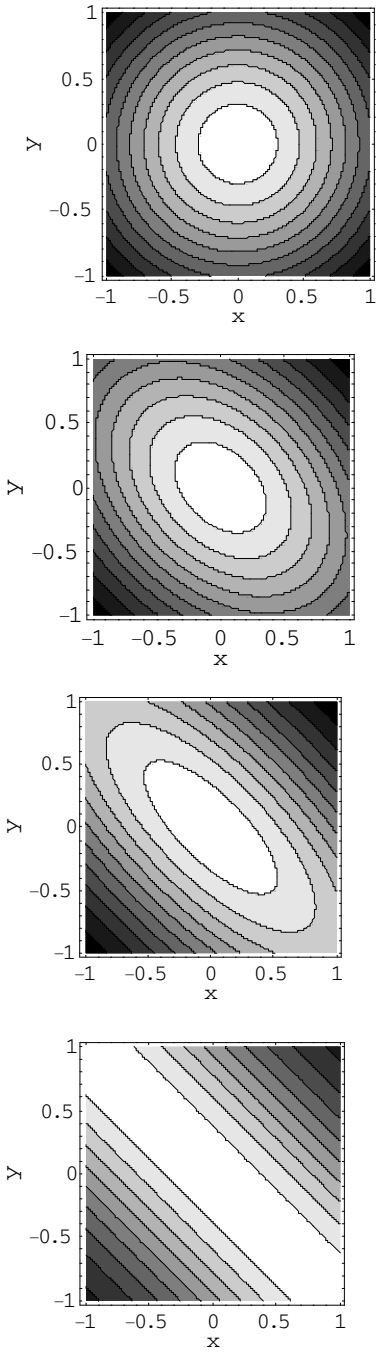

Fig. 1. The ground wavefunctions $\Psi_G^{(N)}$ for the normal phase in the $x-y$ plane as a function of the coupling parameter $\lambda = 0, 0.2, 0.4, 0.527$ with $\omega_0 = 1$, $\omega = 0.9$ and $\kappa = 0.053$.

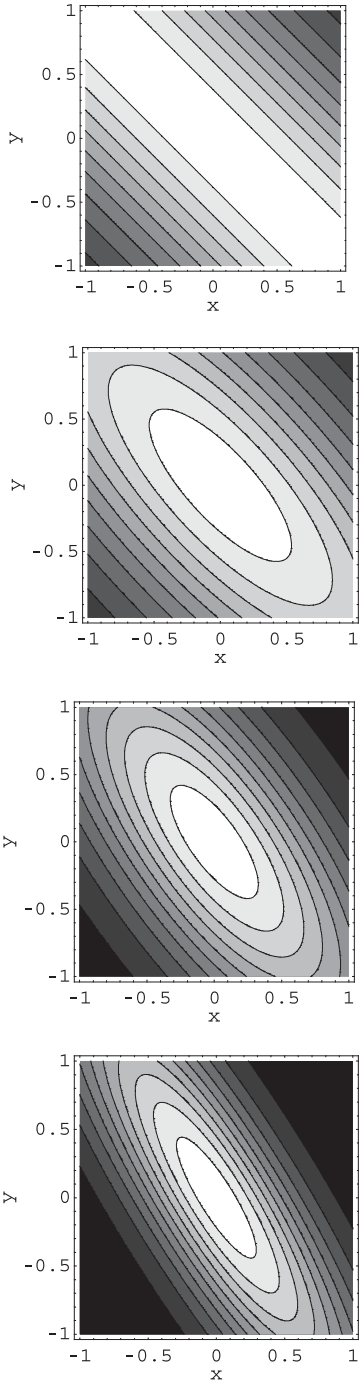

Fig. 2. The ground wavefunctions $\Psi_G^{(S)}$ for the superradiant phase in the $x'-y'$ plane as a function of the coupling parameter $\lambda = 0.5271, 0.6, 0.8, 1$ with $\omega_0 = 1$, $\omega = 0.9$ and $\kappa = 0.053$.

4 The superradiant phase

In order to effectively discuss the superradiant phase we must incorporate the fact that both the atomic ensemble and the field get macroscopic occupations, in which the HP transformation can be also implemented by the following bosonic operators [22]:

$$a^+ \to c^+ + \sqrt{\alpha}, \quad b^+ \to d^+ - \sqrt{\beta}, \tag{12}$$

where $\sqrt{\alpha} = (2\lambda/(\omega + 4\kappa))\sqrt{(j/2)(1-u^2)}$ and $\sqrt{\beta} = \sqrt{j(1-u)}$ with

$$u = \frac{\omega_0(\omega + 4\kappa)}{4\lambda^2} = \frac{\lambda_c^2}{\lambda^2}.$$

This choice of the displacement constants make this quantum system to be fixed at the minimum value of the free energy. With the same calculations as those in the normal

phase Hamiltonian (3) can be written as

$$H^{(S)} = \omega c^+ c + \kappa(c^+ + c)^2 + \frac{\omega_0}{2u}(1+u)d^+ d + \frac{\omega_0(1-u)(3+u)}{8u(1+u)}(d^+ + d)^2 + \lambda u\sqrt{\frac{2}{1+u}}(c^+ + c)(d^+ + d) - j\left[\frac{2\lambda^2}{\omega+4\kappa} + \frac{\omega_0^2(\omega+4\kappa)}{8\lambda^2}\right] - \frac{\lambda^2}{\omega+4\kappa}(1-u), \quad (13)$$

and furthermore can diagonalized, in terms of $c^+ = (\sqrt{\omega}X - i(1/\sqrt{\omega})P_X)/\sqrt{2}$, $d^+ = (\sqrt{\tilde{\omega}}Y - i(1/\sqrt{\tilde{\omega}})P_Y)/\sqrt{2}$ with $\tilde{\omega} = (\omega_0/2u)(1+u)$, as

$$H^{(S)} = \frac{1}{2}\left[(\varepsilon_-^{(S)})^2 Q_1^2 + P_1^2 + (\varepsilon_+^{(S)})^2 Q_2^2 + P_2^2 - \omega - \tilde{\omega}\right] - j\left[\frac{2\lambda^2}{\omega+4\kappa} + \frac{\omega_0^2(\omega+4\kappa)}{8\lambda^2}\right] - \frac{\lambda^2}{\omega+4\kappa}(1-u), \quad (14)$$

where $X = Q_1\cos\gamma^{(S)} + Q_2\sin\gamma^{(S)}$ and $Y = -Q_1\sin\gamma^{(S)} + Q_2\cos\gamma^{(S)}$ with $\tan(2\gamma^{(S)}) = 4\lambda\sqrt{u\omega\omega_0}/[\omega_0^2/u^2 - (\omega^2+4\kappa\omega)]$,

$$\left(\varepsilon_\pm^{(S)}\right)^2 = \frac{1}{2}\left\{\frac{\omega_0^2}{u^2} + (\omega^2+4\kappa\omega) \pm\sqrt{\left[\frac{\omega_0^2}{u^2} - (\omega^2+4\kappa\omega)\right]^2 + 16\lambda^2 u\omega\omega}\right\}. \quad (15)$$

The excitation energy $\varepsilon_-^{(S)}$ is required to remain real so that $\lambda \succeq \lambda_c$.

The ground state energy in this phase is obtained by

$$E_G^{(S)} = -N\left[\frac{2\lambda^2}{\omega+4\kappa} + \frac{\omega_0^2(\omega+4\kappa)}{8\lambda^2}\right]. \quad (16)$$

The corresponding ground state wavefunctions are also diagonal in the $Q_1 - Q_2$ representation and can be expressed by

$$\Psi_G^{(S)}(Q_1, Q_2) = G_-^{(S)}(Q_1)G_+^{(S)}(Q_2), \quad (17)$$

where $G_\pm^{(S)} = (\varepsilon_\pm^{(S)}/\pi)^{\frac{1}{4}}\exp(-(\varepsilon_\pm^{(S)}/2)q^2)$. In the $x-y$ representation as

$$\Psi_G^{(S)}(x,y) = G_-^{(S)}\left(\left(x - \sqrt{2\alpha/\omega}\right)\cos\gamma^{(S)} - \sqrt{\omega_0/\tilde{\omega}}\left(y + \sqrt{2\beta/\omega_0}\right)\sin\gamma^{(S)}\right) \times G_+^{(S)}\left(\left(x - \sqrt{2\alpha/\omega}\right)\sin\gamma^{(S)} + \sqrt{\omega_0/\tilde{\omega}}\left(y + \sqrt{2\beta/\omega_0}\right)\cos\gamma^{(S)}\right). \quad (18)$$

This expression contains displacements involving the macroscopic quantities α and β and can be removed in the new coordinates x' and y' given by $x' = x - \Delta_x$ and $y' = y + \Delta_y$ with $\Delta_x = \sqrt{2\alpha/\omega}$ and $\Delta_y = \sqrt{2\beta/\omega_0}$. The relationship between the coordinate systems $x' - y'$ and $X - Y$ is given simply by $x' = X$ and $y' = \sqrt{\omega_0/\tilde{\omega}}Y$. Furthermore, the coordinate system $x' - y'$ is very useful because although $X - Y$ is the diagonal representation for the superradiant phase, the definition of these coordinates depends upon $\tilde{\omega}$ and hence upon the parameter κ. By means of these coordinates, the ground state wavefunctions can be rewritten as

$$\Psi_G^{(S)}(x',y') = \left(\frac{\omega_0}{\tilde{\omega}}\right)^{\frac{1}{4}} G_-^{(S)}\left(x'\cos\gamma^{(S)} - \sqrt{\omega_0/\tilde{\omega}}y'\sin\gamma^{(S)}\right) \times G_+^{(S)}\left(x'\sin\gamma^{(S)} + \sqrt{\omega_0/\tilde{\omega}}y'\cos\gamma^{(S)}\right). \quad (19)$$

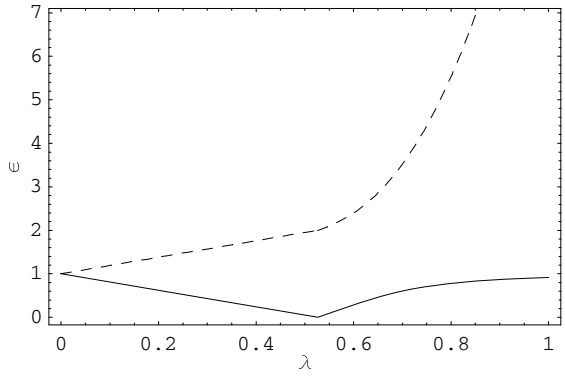

Fig. 3. The excitation energies of the photonic and atomic mode as a function of the coupling parameter λ with $\omega_0 = 1$, $\omega = 0.9$ and $\kappa = 0.053$.

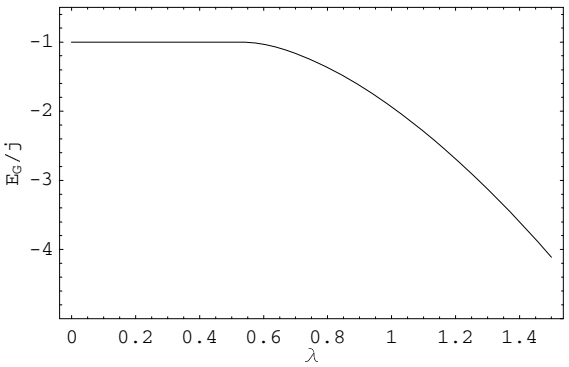

Fig. 4. The scaled ground state energy E_G as a function of the coupling parameter λ with $\omega_0 = 1$, $\omega = 0.9$ and $\kappa = 0.053$.

Figure 3 shows that the ground state wavefunctions are varied via the coupling parameter λ from 0.5271 to 1 with $\omega_0 = 1$, $\omega = 0.9$ and $\kappa = 0.053$. It can be seen that the main results are contrary to those in the normal phase.

5 Quantum phase transition

Having obtaining the effective Hamiltonians of the normal and superradiant phases in the thermodynamical limit $j \to \infty$, we now reveal some properties for this quantum system of its two phases. The numerical results of the excitatation energies for the photonic and atomic mode and

the scaled ground state energy are respectively shown in Figures 3 and 4. From Figure 3 we can see that as coupling approaches the critical transition point λ_c, the excitation energy of photonic mode vanishes, which proves the existence of QPT. In contrast, the excitation energy of atomic mode tends towards a value of $\sqrt{\omega_0^2 + (\omega^2 + 4\kappa\omega)}$ as $\lambda \to \lambda_c$ from either direction. The scaling behavior of the excitation energy of the photonic mode is seen to be $\varepsilon_- \propto |\lambda - \lambda_c|^{2\mu}$ with the exponent $\mu = 1/4$ describing the divergence of the characteristic length $\xi = (\varepsilon_-)^{-1/2}$. Figure 4 shows that in the normal phase there is no interaction between the atoms and field and all atoms are located at the ground state, however, this interaction occurs in the superradiant phase which implies that both the atoms and field acquire the macroscopic excitations. It can be also checked that the second-order derivative of the scaled ground state energy with respect to λ has discontinuity which displays the existence of QPT. At the phase transition point λ_c, the Hamiltonian for both the normal and superradiant phases is equal and can be written as $H^{(N)}(\lambda_c) = H^{(S)}(\lambda_c) = \{[\omega_0^2 + (\omega^2 + 4\kappa\omega)]q_2^2 + p_2^2 - \omega - \omega_0\}/2 - j\omega_0$. Comparing our results with the previous known results given in the standard Dicke model it can be found that the critical transition point is increased due to the existence of the form A^2.

6 Conclusions

In conclusion, we have presented the analytical discussions of QPT for generalized SMSM with the form A^2. By means of the HP transformation the ground state energies and wavefunctions of both the normal and superradiant phases have been obtained and therefore the scaling behavior near the critical transition point λ_c has been revealed.

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